Observables of $D = 4$ Euclidean Supergravity and Dirac Eigenvalues

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The observables of supergravity are defined to be the phase space gauge invariant objects of the theory. Dirac eigenvalues are obsevables of Euclidean gravity on a compact manifold [1,2,3]. We resume the basic results for supergravity [4,5], [6,7].

The general setting which we consider is of minimal supergravity in four dimensions on a compact spin manifold without boundary endowed with an Euclidean metric $g_{\mu\nu}(x) = e^a_\mu(x)e_{\nu a}(x)$ where the indices of the tetrad $e^a_\mu(x)$ are spacetime indices $\mu = 1, \cdots, 4$ and internal Euclidean indices $a = 1, \cdots, 4$, respectively. They are raised and lowered by the Euclidean metric δ_{ab}. The gravitino is represented by a Euclidean spin-vector field $\psi_\mu(x)$ and it should be defined by a modified Majorana condition, since the group $SO(4)$ does not admit Majorana spinors. A standard condition is $\bar{\psi} = \psi^T C$ [8] (see also [9,10]).

By definition, the phase space of the theory is the space of the solutions of the equations of motion, modulo the gauge transformations which are: diffeomorphisms in four dimensions, local $SO(4)$ rotations and the local $N = 1$ supersymmetry. The phase space is covariantly defined and its elements are all pairs (e, ψ) that are solutions of the equations of motion modulo gauge transformations. Therefore, it is sufficient to consider only on-shell supersymmetry and the supersymmetric algebra closes over graviton and gravitino. The observables of the theory are the functions on the phase space.

The self-adjoint Dirac operator and the spin connection are defined as follows

$$D = i\gamma^a e^\mu_a(\partial_\mu + \omega_{\mu bc}(e,\psi)\gamma^b\gamma^c) \quad , \quad \omega_{\mu ab}(e,\psi) = \overset{\circ}{\omega}_{\mu ab}(e) + K_{\mu ab}(\psi) \qquad (1)$$

where $\overset{\circ}{\omega}_{\mu ab}(e)$ is the usual spin-connection of gravity. The following relations hold

$$\overset{\circ}{\omega}_{\mu ab}(e) = \frac{1}{2}e_a^\mu(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) + \frac{1}{2}e_a^\rho e_b^\sigma \partial_\sigma e_{\rho c} e_\mu^c - (a \leftrightarrow b)$$

$$K_{\mu ab}(\psi) = \frac{i}{4}(\bar{\psi}_\mu \gamma_a \psi_b - \bar{\psi}_\mu \gamma_b \psi_a + \bar{\psi}_b \gamma_\mu \psi_a). \tag{2}$$

The chosen topology of spacetime manifold guarantees that the spectrum of the Dirac operator is discrete

$$D\chi^n = \lambda^n \chi^n. \tag{3}$$

λ^n's define a discrete family on the space of all gravitons and gravitinos and should be gauge invariant objects. A general argument is that the eigenvalues of a general covariant operator should be also invariant. By checking up the invariance of λ^n's explicitly one obtains a set of equations that should be satisfied by the pairs (e, ψ).

The transformation of the fields under gauge transformations are given by the following relations

$$\delta e_\mu^a = \xi^\nu \partial_\nu e_\mu^a \quad , \quad \delta\psi_\mu = \xi^\nu \partial_\nu \psi_\mu, \tag{4}$$

where $\xi = \xi^\mu \partial_\mu$ is an infinitesimal vector field,

$$\delta e_\mu^a = \theta^{ab} e_{b\mu} \quad , \quad \delta\psi_\mu^\alpha = \theta^{ab}(\sigma_{ab})_\beta^\alpha \psi_\mu^\beta, \tag{5}$$

where $\theta_{ab} = -\theta_{ba}$ parametrize an infinitesimal $SO(4)$ rotation and $\sigma^{ab} = i\Sigma^{ab}$, and

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu \quad , \quad \delta\psi_\mu = \mathcal{D}_\mu \epsilon, \tag{6}$$

where $\epsilon(x)$ is an infinitesimal Majorana spinor field, i. e. it obeys the Majorana conjugation given above. If λ^n's are invariant under all gauge transformations the following relations must hold

$$\mathcal{T}_a^{n\mu} \partial_\nu e_\mu^a - \Gamma_a^{n\mu} \partial_\nu \psi_\mu^\alpha = 0, \tag{7}$$

$$\mathcal{T}_a^{n\mu} e_{b\mu} + \Gamma^{n\mu} \sigma_{ab} \psi_\mu = 0, \tag{8}$$

$$\mathcal{T}_a^{n\mu} \bar{\epsilon}\gamma^a \psi_\mu + \Gamma^{n\mu} \mathcal{D}_\mu \epsilon = 0. \tag{9}$$

Here, $\mathcal{T}_a^{n\mu} = T_a^{n\mu} + K_a^{n\mu}$, where $T_a^{n\mu}$ is the "energy-momentum tensor" of the spinor χ^n [2], $K_a^{n\mu} =< \chi^n | i\gamma_a K_{bc}^{\mu}(\psi)\sigma^{bc}|\chi^n >$, and

$$\Gamma_a^{n\mu} = \frac{i}{4}\int \sqrt{e}\chi^{n*}\gamma^a e_a^{\nu}[\bar{\psi}_{\nu}^{\beta}(\gamma_b)_{\alpha\beta}e_c^{\mu} - \bar{\psi}_{\nu}^{\beta}(\gamma_c)_{\alpha\beta}e_b^{\mu} + \bar{\psi}_b^{\beta}(\gamma_{\nu})_{\alpha\beta}e_c^{\nu}]\sigma^{bc}\chi^n. \quad (10)$$

Eq.(5) is a covariant equation, therefore its variation under gauge transformations should cancel. The variation of Eq.(5) under diffeomorphisms, $SO(4)$ rotations and local supersymmetry and the equations (7), (8) and (9) imply the following relations

$$\{[b^{\mu}(\xi) - c(\lambda\xi)^{\mu}]\partial_{\mu} + f(\xi)\}\chi^n = 0, \quad (11)$$

$$[\theta_a^a D - g(\theta) + h(\theta)]\chi^n = 0, \quad (12)$$

$$[j_a^{\mu}(\epsilon)\partial_{\mu} + k_a(\epsilon) + l_a]\chi^n = 0. \quad (13)$$

Here, the following notations are used

$$h(\theta) = i(\lambda^n - D)\theta\sigma \quad , \quad j_a^{\mu}(\epsilon) = \frac{1}{2}\gamma_a\bar{\epsilon}\psi^{\mu} \quad , \quad k_a(\epsilon) = \frac{1}{2}\gamma_a\bar{\epsilon}\psi^{\mu}\omega_{\mu cd}\sigma^{cd} \quad (14)$$

$$b^{\mu}(\xi) = i\gamma^a b_a^{\mu}(\xi) \quad , \quad b_a^{\mu}(\xi) = \xi^{\nu}\partial_{\nu}e_a^{\mu} - e_a^{\nu}\partial_{\nu}\xi^{\mu} - 2e_a^{\nu}\xi^{\mu}\omega_{\nu bc}\sigma^{bc} \quad (15)$$

$$c(\lambda,\xi)^{\mu} = (\lambda^n - D)\xi^{\mu} \quad , \quad f(\xi) = i\gamma^a\xi^{\nu}\partial_{\nu}(e_a^{\mu}\omega_{\mu bc})\sigma^{bc}, \quad (16)$$

$$g(\theta) = [\gamma^c e_c^{\mu}([\theta\sigma, \omega_{\mu ab}] - \partial_{\mu}\theta\sigma M_{ab})]\sigma^{ab}, \quad (17)$$

$$l_a = e_a^{\mu}[B_{\mu cd} - \frac{1}{2}e_{\mu d}B_{ec}^{e} + \frac{1}{2}e_{\mu c}B_{ed}^{e}]\sigma^{cd}. \quad (18)$$

We interpret the relations (7), (8), (9) and (11), (12), (13) as two sets of constraints on the phase space of the theory. By construction, the second set is determined by the first one. An analysis of the linear dependency of these constraints in view of the quantization of the theory was performed in [4,5,6,7] where some other aspects of the theory were discussed. A "spectral formulation" of supergravity is not known at present.

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